Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled-map lattice

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We study the long-wavelength properties of a two-dimensional lattice of chaotic coupled maps, in which the dynamics has Ising symmetry. For sufficiently strong coupling, the system orders ferromagnetically. The phase transition has static and dynamic critical exponents that are consistent with the Ising universality class. We examine the ordered phase of the model by analyzing the dynamics of domain walls, and suggest that the dynamics of these defects allow a complete characterization of the long-wavelength properties of this phase. We argue that, at large length scales, the correlations of this phase are precisely those of an equilibrium Ising model in its ordered phase. We also speculate on what other phases might occur in more general such models with Ising symmetry.

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INTRODUCTION

Systems of coupled chaotic maps have lately received special attention, because of the possibilities they offer for a greater understanding of *dissipative* spatially extended systems with many degrees of freedom [1]. In particular, one might hope to achieve a description of long-wavelength, long-time phenomena, analogous to those which we have acquired for equilibrium systems. There, thermodynamics and equilibrium statistical mechanics identify for us a small set of quantities that characterize what are believed to be the essential long-wavelength features of an interacting, many-body systems. Central to developing such a "hydrodynamic" description of a system is the observation that its configuration space and dynamics may be very complicated on a microscopic level; nevertheless, once we coarse grain the system and look only at appropriate averages, the number of degrees of freedom is effectively reduced, from $O(N)$, where N is the number of microscopic degrees of freedom, to just those slow modes, conserved fields, or order parameters that participate in the long-wavelength, long-time dynamics.

One may ask whether a similar description of extended nonequilibrium dissipative systems is possible. An advantage of coupled chaotic maps in studying this question is that we may readily control microscopic properties of the system, by choosing appropriate maps and couplings, to determine the order parameters that will be relevant at large scales. Provided the interactions are *local,* one expects that nontrivial long-wavelength properties at or near the steady state will reflect either conservation laws or symmetries of the system.

An essential feature of a long-wavelength, coarse-grained description of a deterministic chaotic dynamics is an understanding of how the *local* chaos couples to the long-wavelength dynamics, a question that has been discussed recently by a number of authors [1,2]. The coarse-grained dynamics consists of a *deterministic* local dynamics of the long-wavelength modes, plus a noiselike term due to forcing of long-wavelength modes by chaotic short-wavelength modes ("local chaos") that are not explicitly included in the coarse-grained description. Zaleski [2] considers a partial differential equation in one space and one time dimension, the Kuramoto-Sivashinsky (KS) equation. He argues that its long-wavelength dynamics is statistically identical to a stochastically driven Burgers' equation. This implies that the local chaos in the KS equation decorrelates from the long-wavelength dynamics, thus acting like a stochastic noise source. Bourzutschky and Cross [1] (BC) examine a discrete space-time system: a lattice of coupled chaotic maps whose local dynamics yield a conservation law. Once again, it emerges that the correlations of short-wavelength chaos with long-wavelength dynamics are unimportant: Local chaos can be modeled as stochastic noise. The general validity of this observation in a hydrodynamic description of a chaotic system is, we believe, an interesting question.

A natural approach to characterizing nonequilibrium systems would be to extend concepts useful in the equilibrium case, as suggested by Hohenberg and Shraiman [3]. The work of BC suggests, for example, that their coupled-map lattice develops a "chemical potential" for the conserved quantity. In a related effort to discern the applicability of statistical-mechanical reasoning to nonequilibrium systems, Shraiman *et al.* [4] study the complex Ginzburg-Landau partial differential equation in two space-time dimensions. The "order parameter" is a complex scalar field and has a continuous symmetry under rotations about the origin in the complex plane. The symmetry suggests the importance of *defects* ("space-time vortices") in characterizing the dynamics, and Shraiman *et al.* present numerical evidence that one may characterize the chaotic phases of this model by the presence or absence of defects.

An additional basic question involves the relevance of the Gibbs free energy to a dynamical system with many degrees of freedom. A free energy is ordinarily expected to describe a system that satisfies detailed balance: the rate of transition from *any* state of the system to any other state coincides with the rate of the inverse transition. Most often, arguments for the applicability of a Gibbs ensemble rely upon the assumption of detailed balance; its absence is ordinarily offered as *ipse facto* evidence for the nonequilibrium character of the system in question. On the other hand, it is generally appreciated that detailed balance is not *necessary* for a Gibbs-like description to apply; all that is required is that some coarse-grained quantities ψ_i obey a Langevin dynamics [5]:

$$\frac{\partial\psi_i}{\partial t}=-\sum M_{ij}\frac{\partial\mathcal{F}}{\partial\psi_j}+\zeta_i \ , \tag{1}$$

where

$$\langle\zeta_i(\mathbf{r},t)\zeta_j(\mathbf{r}',t')\rangle=2\Omega_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \ , \tag{2}$$

Ω_{ij} is the symmetric part of M_{ij}, and $\mathcal{F}(\psi_i)$ denotes the *reduced free energy* (i.e., the free energy divided by the temperature). We feel that it should be of some interest to understand when it is to be expected that an *irreversible* dynamics, in particular a dynamics that does not satisfy detailed balance, nevertheless satisfies (1) and (2) for nontrivial coarse-grained quantities.

A natural candidate for an interesting coarse-grained dynamics is the dynamics of *defects.* A recent proposal for the dynamics of turbulent fluids at wavelengths large compared to some microscopic cutoff relies on an assumption of equilibrium, and the corresponding defects-vortices play a central role in this theory [6]. Other workers have also studied the complex Ginzburg-Landau equation, with an emphasis on the statistical characterization of the defect density and its fluctuations [7].

In our work, we focus on these general issues by looking at a simple, discrete space-time coupled-map system with a discrete symmetry: a nonequilibrium, deterministic, and chaotic analog of the two-dimensional Ising model. We argue that the symmetry enables us to characterize the possible phases in this model, and in a special phase permits a *quantitative* description of the long-wavelength dynamics of the system. The defects we study in the ordered phase are the domain walls and *droplets* of flipped "spins" that represent the dominant long length-scale fluctuations, much as in the equilibrium Ising model. We argue that the long-wavelength dynamics of the domain walls in one ordered phase of our coupled-map system is actually identical to that of an *equilibrium* Ising model.

DROPLET MODELS

The nontrivial character of the ordered phase of the *equilibrium* two-dimensional Ising model has been well established [8], but is generally unappreciated. In particular, the equilibrium model displays certain *universal* features that follow only from the Ising symmetry of the microscopic Hamiltonian. Since we shall apply to our nonequilibrium system arguments analogous to those used for the equilibrium Ising model, we briefly review, in this section, the way droplets determine long-wavelength correlations in the low-temperature phase of the Ising model, in the absence of a magnetic field or any conservation laws.

Defect dynamics in the two-dimensional Ising model may be described by a reduced free-energy functional $\mathcal{F}$:

$$\mathcal{F}(\mathbf{X})=\int\tau(\theta(\mathbf{X}(s)))ds \ , \tag{3}$$

where $\mathbf{X}(s)$ represents the positions of the domain walls in the system, as parametrized by arc length s. The integral on ds is over all domain walls in the system. θ denotes the angle of the local normal to the domain wall at $\mathbf{X}$ with respect to the lattice axes, and $\tau(\theta)$ denotes the reduced *surface tension.* The surface tension may be computed from a microscopic Hamiltonian, but here we take it as a phenomenological parameter. We assume spatial inversion is a symmetry of the system, so $\tau(\theta)=\tau(\theta+\pi)$. By means of the Wulff construction [9], $\tau(\theta)$ generates an item of physical interest, the *equilibrium crystal shape.*

An elementary variational calculation yields a Langevin equation of the form (1) for the domain-wall dynamics:

$$\frac{dX_n}{dt}=-\Gamma(\theta)\{\sigma(\theta)\kappa(s)\}+\nu(\theta,s,t) \ , \tag{4}$$

where we have introduced a local, δ-function-correlated, stochastic noise, ν, of amplitude $\Gamma(\theta)$:

$$\langle\nu(\theta,s,t)\nu(\theta',s',t')\rangle=\Gamma(\theta)\delta(s-s')\delta(t-t') \ , \tag{5}$$

X_n denotes the normal component of the domain-wall coordinate, the reduced *surface stiffness* $\sigma(\theta)$ is given by

$$\sigma(\theta)=\tau(\theta)+\frac{d^2\tau(\theta)}{d\theta^2} \ , \tag{6}$$

and $\kappa(s)$ denotes the curvature of the domain wall at s.

In the ordered phase of a model governed by the free energy $\mathcal{F}$, the spin-spin correlation functions acquire a nontrivial character determined by the droplet fluctuations. The droplet fluctuations play a special role because the only way two well-separated spin (i.e., spins with a spatial or temporal separation much larger than the correlation length or time, respectively) may develop any correlation is when they are both contained in the *same* droplet [8]. In this way, the broken symmetry is reflected in the low-temperature phase by *anomalous decay laws* for the correlations. For example, the long-distance, equal-time correlation between spins y behaves as [8]

$$\langle y(\mathbf{x}_1)y(\mathbf{x}_2)\rangle_c\propto\exp\{-2\tau(\theta)r\}/r^2 \ , \tag{7}$$

where the subscript c refers to the connected part of the correlation function; $r=|\mathbf{x}_1-\mathbf{x}_2|$, and θ denotes the angle between $\mathbf{x}_1-\mathbf{x}_2$ and the lattice axes.

In summary, it is the properties of the antiphase droplets that determine the long-distance or long-time hydrodynamic behavior of the ordered phase of the two-dimensional Ising model. These droplet properties, in turn, are traditionally derived from the fundamental starting point of a Hamiltonian energy functional for the model [8]. We shall argue that one may in fact view the droplets *themselves* as fundamental, giving rise to the features described above even in the *absence* of a Hamiltonian energy for the system.

THE MODEL

The dynamical system we study consists of a two-dimensional (2D) square lattice of coupled, chaotic scalar maps. The map $\phi(y)$, which we require to be the same at every site, has at least two essential properties: (1) chaos, that is, a single uncoupled map has a positive Lyapunov exponent; and (2) odd parity, $\phi(-y)=-\phi(y)$. The map that has received most of our attention is a tent map, illustrated in Fig. 1:

$$\phi(y)=\begin{cases}-2-3y & \text{for } -1\le y\le -\frac{1}{3}\\ 3y & \text{for } -\frac{1}{3}\le y\le \frac{1}{3}\\ 2-3y & \text{for } \frac{1}{3}\le y\le 1\ .\end{cases} \tag{8}$$

The discrete-time dynamics of the coupled system is

$$y_i^{t+1}=\phi(y_i^t)+g\sum_j{}^i\{\phi(y_j^t)-\phi(y_i^t)\}\ , \tag{9}$$

where the superscript indexes the time, the subscript the spatial coordinate, and g is the coupling constant. The sum is over all four nearest neighbors, j, of site i. We require that the real numbers y_i at each site take only values in the interval $[-1,1]$, and to maintain this condition we restrict the coupling to the range $0\le g\le\frac{1}{4}$. We do not add any external noise, although we shall later briefly discuss the effect such a term might have.

For $g=0$, it is clear that this array of maps is chaotic, and in fact it is ergodic on the full configuration space $[-1,1]^N$, where N is the number of sites, visiting all configurations with uniform probability in its long-time statistical steady state. For arbitrary g, if all sites have precisely the same value of y they will remain the same; however, these uniform states are unstable and are of zero measure in the full configuration space.

For $g>0$, the coupling induces correlations in the statistical steady state, which may be studied numerically. In particular, we find that the array has a *ferromagnetically ordered* steady state for $0.2054\lesssim g\lesssim 0.24$. That is, in this ferromagnetic phase the long-distance correlations $\langle y_i y_k\rangle$, averaged in the steady state, are nonzero in the limit of large separations between points i and k. In the paramagnetic phase the statistical steady state is apparently independent of the initial condition for initial conditions chosen randomly on $[y_{\min},y_{\max}]^N$, with $-1\le y_{\min}<y_{\max}\le 1$. In the ferromagnetic phase for large arrays this attractor effectively splits into two equivalent attractors, one with positive and one with negative magnetization. The frequency of transitions between these two attractors then appears to vanish exponentially with increasing array size.

We have checked that the array dynamics is chaotic for couplings in both the paramagnetic and ferromagnetic regimes, by examining the largest Lyapunov exponent, $\lambda_{\max}(g)$, determined by the rate of divergence of two initially nearby configurations. For vanishing coupling, the independent site dynamics of (8) obviously yields $\lambda_{\max}(0)=\ln 3$, whereas for larger coupling $\lambda_{\max}$ varies continuously with g, always remaining larger than 0.5. Thus, our array undergoes an ordering transition between two chaotic phases. The paramagnetic phase *per se* being presumably of little interest at long wavelengths, the remainder of the paper primarily concerns ordered phases and phase transitions.

A CRITICAL POINT

The immediate question arises as to the universality class of the phase transition between the paramagnetic and ferromagnetic phases discussed above. Bennett and Grinstein [10] and Grinstein, Jayaprakash, and He [11] have argued that the ferromagnetic critical point of a translationally invariant system with Ising symmetry generically ought to fall within the Ising universality class for both statics and dynamics. Their argument relies on the assumption of a Langevin description for the

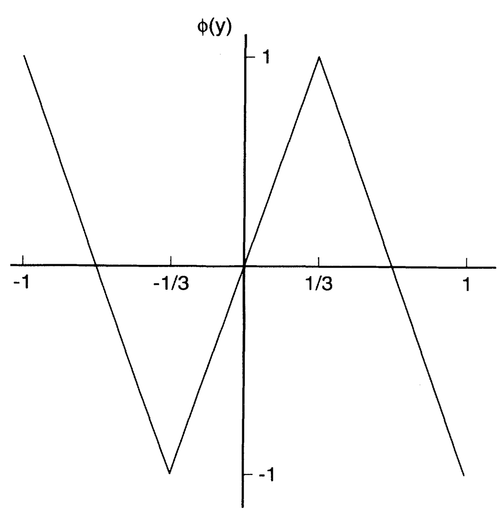

FIG. 1. The function $\phi(y)$ of Eq. (8) that represents the tent map studied in the bulk of this paper.

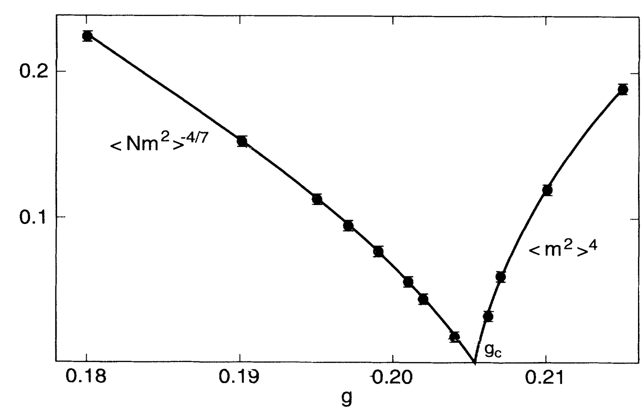

FIG. 2. The mean-square magnetization, scaled by system size for $g<g_c$ and raised to the appropriate power $[-\frac{4}{7}=-1/\gamma$ for $g<g_c$, and $4=1/(2\beta)$ for $g>g_c]$ so that the quantities plotted should vanish with a finite, nonzero slope at the critical point g_c if the phase transition is in the Ising universality class. The smooth curves show that these data are consistent with this hypothesis.

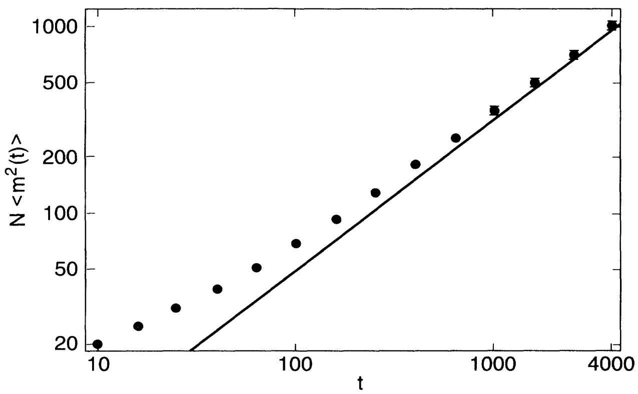

FIG. 3. Growth of the mean-square magnetization with time at $g = g_c \simeq 0.2054$, after starting in a random initial state. The straight line is drawn with slope $\gamma/z\nu \simeq 0.81$, the long-time growth exponent for this quantity at the critical point of the 2D Ising model.

coarse-grained dynamics:

$$\dot{y}_i = Q_i(\{y_j\}) + \eta_i(\{y_i\}, t) , \tag{10}$$

where y_i is the local field corresponding to coarse-grained spins, Q describes the deterministic part of the dynamics, and η represents the noise. Grinstein, Jayaprakash, and He [11] argue that contributions to Q that represent local interactions, but cannot be derived from a Hamiltonian, are *irrelevant* at the critical point (within a renormalization-group ϵ expansion around $d = 4$), as are *local* correlations of the noise with the state of the system. Bennett and Grinstein verify this claim for a dynamics subject to external noise; in our system the noise is due to the deterministic local chaos.

We have checked that the dynamics of our array yields a phase transition at $g_c \simeq 0.2054$ consistent with the two-dimensional Ising universality class and the arguments of Refs. [10] and [11]. We measured the "magnetization" density $m \equiv (1/N)\sum_i \mathrm{sgn}(y_i)$ in numerical simulations of the coupled-map system, where N is the number of sites of a square lattice with periodic boundary conditions. The mean-square magnetization in the steady state, $\langle m^2 \rangle$, varies in a manner consistent with the appropriate power of $|g - g_c|$ expected from the 2D Ising universality class, on both sides of the transition (Fig. 2). To confirm the dynamical universality, we have evaluated the growth of $\langle m^2(t) \rangle$ with time for the critical coupling, $g = 0.2054 \simeq g_c$, starting from a random initial configuration. As displayed in Fig. 3, the long-time growth exponent is once again consistent with Ising universality. Here one sees that deviations from the asymptotic behavior are quite apparent at early time and thus short length scales. We also examined the finite-size scaling of $\langle m^4 \rangle / \langle m^2 \rangle^2$ near g_c, and again observe behavior fully consistent with Ising universality for large enough sizes.

AN ORDERED PHASE

Whereas the arguments discussed in the previous section [10,11] suggest that we understand quite generally the critical point of this kind of Ising system (provided the Langevin description obtains), we have until now achieved no such understanding of the ordered phase. In the special case that a local dynamics is (microscopically) reversible, detailed balance implies that the system is in fact governed by the free energy of a generalized Ising model: that is, by some local spin Hamiltonian with Ising symmetry [11]. The array is then a "kinetic Ising model," a type of model that has received much study [12]. Our model does not fall into this special category, since there is no reason for our dynamics to be reversible at small scales. Indeed, detailed balance entails that for *any* coarse graining of system configurations, the number of transitions from one state to another, per unit time, coincides with the number of transitions in the opposite direction. There is no reason for the single-site map ϕ to obey such a condition in the absence of coupling, and we have numerically verified, for a variety of coarse grainings involving small numbers of sites, that the *small-scale* dynamics deviates substantially from detailed balance of its transition rates.

Nevertheless, we may characterize the *large-scale* dynamics in our more general situation. The dynamics of domains and domain walls (interfaces) provide a natural route to a long-wavelength description of the ordered phase. Once we coarse grain over small distances (at least of order of the correlation length), domains and sharp interfaces are the only features that survive. Just as in the equilibrium case one finds that droplet fluctuations, whose dynamics are governed by an interface Hamiltonian, provide the *only* contribution to long-wavelength properties; we claim that a droplet picture, based solely on domains and their fluctuations, yields a *quantitative* description of long-wavelength properties.

Three components comprise our argument: (1) a Langevin description of the *interface* dynamics is valid at long wavelengths; (2) this Langevin description is precisely that of the interface dynamics of some generalized Ising model; (3) large droplet fluctuations do in fact spontaneously occur in the ordered phase of our model. We conclude that the long-wavelength properties of the ordered phase of this model are precisely those of an (*equilibrium, generalized*) Ising model. However, we will argue below that within the more general class of coupled maps with Ising symmetry, there exist, in addition, other regimes with interfaces, in which some or all of the above three conditions are violated.

A. Domain-wall dynamics

To understand the general case, it is useful to examine a particular coarse graining of the dynamics: the Langevin dynamics of interfaces, or domain walls. Even in the absence of defects generated spontaneously by the dynamics of the array (that is, even when droplets are not spontaneously nucleated), it is nevertheless possible to inquire about the dynamics of domain walls, by applying appropriate boundary conditions. In particular, we apply periodic and antiperiodic boundary conditions in orthogonal directions, to establish an interface whose mean angle θ with respect to the lattice axes may be fixed at an arbitrary value (Fig. 4). The ferromagnetically ordered domains on either side of the interface have opposite magnetizations. Given such boundary conditions, we apply a Langevin description to the fluctuations of the interface about its mean position. It is straightforward to write down a Langevin equation for the (time-dependent) deviation of the interface from a straight reference position, the fluctuations of which we denote by $h(x,t)$

$$\frac{\partial h}{\partial t}=\sigma_\theta \nabla^2 h+\eta_\theta(x,t) , \tag{11}$$

where σ_θ is a constant describing interfacial resistance to bending, and η_θ is a stochastic noise arising from the local chaos. This equation consists of the lowest-order contributions invariant under lattice translation and inversion, and spin inversion $y \leftrightarrow -y$. Dimensional analysis shows that any further local interactions or local correlations between noise and interface shape are irrelevant at long wavelengths, provided the noise has only short-range temporal and spatial correlations.

This Langevin description may be checked by numerical examination of the scaling of mean-square interface displacement with interface length following from (11), which is well known for Ising models [8]:

$$\langle [h(0)-h(x)]^2 \rangle \propto x , \tag{12}$$

for large distances x along the interface. We have checked this scaling for the ordered phase of our coupled-map lattice for several boundary conditions, angles, and microscopic algorithms for finding the interface [13]. Our results are consistent with Eq. (12).

B. Droplet dynamics

We may extend this kind of analysis to the dynamics of droplets, or connected regions of inverted (coarse-grained) spins surrounded by a closed interface. We examine length scales where the boundary of a droplet is well described by a (one-dimensional) curve. The results of the preceding subsection, showing that the amplitude of the transverse interface fluctuations grows only as the square root of the interface length, suggests that such length scales are attained by sufficiently large droplets. We use a local coordinate parametrization where $\mathbf{X}$ is a position vector giving coordinates of the curve parametrized by s and $\mathbf{n}$ is the normal to the curve. We may describe the curve completely within this local parametrization by means of $\theta(s,t)$ and its derivatives $d\theta/ds, d^2\theta/ds^2, \ldots$, where θ represents the angle of $\mathbf{n}$ with respect to the direction of the lattice axes, and s is the distance along the curve so $|d\mathbf{X}/ds|=1$. It emerges, again by a dimensional argument, that derivatives of θ with respect to s higher in order than the first are irrelevant at long wavelengths. In terms of the curvature, $\kappa \equiv d\theta/ds$, we write a Langevin equation for the motion of the curve:

$$\frac{d\mathbf{X}}{dt}=f(\theta,\kappa)\mathbf{n}+\eta(\theta,\kappa,s,t)\mathbf{n} , \tag{13}$$

where f represents the deterministic part of the coarse-grained dynamics, and η noise from the local chaos. (Motion of the curve in the direction of the tangent corresponds merely to reparametrization.) We next assume small curvature to expand the dynamics and noise in powers of the curvature; powers larger than the first are irrelevant. Lattice inversion symmetry enables us to write

$$f(\theta,\kappa)=\zeta(\theta)\kappa . \tag{14}$$

Symmetry under spin inversion requires the distribution of the noise η to be an even function of κ, yielding a noise term with no dependence on curvature to relevant order. Our Langevin equation then becomes

$$\frac{d\mathbf{X}}{dt}=\{\zeta(\theta)\kappa+\eta(\theta,s,t)\}\mathbf{n} . \tag{15}$$

Assuming the effective noise has only short-distance correlations, we have

$$\langle \eta(\theta,s,t)\eta(\theta',s',t')\rangle=2\Gamma^2(\theta)\delta(s-s')\delta(t-t') , \tag{16}$$

with $\Gamma^2(\theta)$ the angle-dependent noise intensity. Under rescaling of space and time $t\to b^2 t$; $\mathbf{X}\to b\mathbf{X}$, we obtain

$$\frac{d\mathbf{X}}{dt}=\left\{\zeta(\theta)\kappa+\frac{\eta}{b^{1/2}}\right\}\mathbf{n} , \tag{17}$$

which corresponds to the expected relative scaling of the dynamics and noise in the 2D Ising model.

The validity of the Langevin description may be tested further by examining the dynamics of the decay of large droplets. We introduce large droplets by inverting a convex domain of spins in a lattice that has relaxed to its steady state, and observe their decay in time. The Langevin description entails that the mean area of a large droplet, D, vary linearly with time:

$$\langle D(0)-D(t)\rangle \sim t . \tag{18}$$

Once again, we have observed this behavior numerically

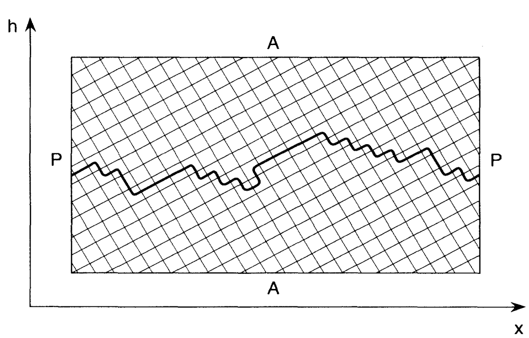

FIG. 4. The geometry for the measurement of the mean-square interface height difference as a function of the interface length. The boundaries of the rectangular piece of lattice studied are oriented at a chosen angle with respect to the lattice axes. The two sides are joined by periodic boundary conditions (P), while the top and bottom are joined by antiperiodic boundary conditions (A). In the ordered phase this forces the interface to be present.

(Fig. 5). As the droplets diminish to a characteristic radius comparable to the correlation length, we no longer expect (18) to apply, yielding, for small deviations from the mean magnetization, the saturation evident in Fig. 5.

In fact, the Langevin description (15) is precisely that of an *equilibrium* Ising model [8]. We may exhibit this identification by factoring out the angular dependence of the noise in the Langevin equation (15):

$$\frac{d\mathbf{X}}{dt}=\Gamma(\theta)\left\{\frac{\zeta(\theta)}{\Gamma(\theta)}\kappa+\nu(s,t)\right\}\mathbf{n}\ , \tag{19}$$

where $\Gamma(\theta)$ is defined as in Eq. (16), and the noise ν is δ-function correlated in s and t, but now is intensity has no angular dependence. Defining the reduced stiffness $\sigma(\theta)\equiv\zeta(\theta)/\Gamma(\theta)$, we may rewrite (15) as

$$\frac{dX_n}{dt}=\Gamma(\theta)\left\{-\frac{\delta\mathcal{F}}{\delta\mathbf{X}}+\nu\right\}, \tag{20}$$

where the reduced free energy $\mathcal{F}$ is defined as in (3). $\Gamma(\theta)$ is a kinetic coefficient.

While (3) is the coarse-grained free energy of an Ising model, the possibility remains that the reduced surface tension $\tau(\theta)$ may exhibit singularities as a function of θ, or be nonpositive, neither of which occurs in a two-dimensional Ising model with short-range interactions, except at zero temperature. We have estimated $\sigma(\theta)$ for several θ by examining the amplitude of interface fluctuations in the geometry of Fig. 4. We have used a variety of methods for measuring the interface position, both including and neglecting overhangs; all methods yield (reduced) stiffnesses which agree quantitatively. Furthermore, the relative stiffnesses appear to increase smoothly and monotonically as θ varies from 0 to $\pi/4$, consistent with both $\sigma(\theta)$ and $\tau(\theta)$ being smooth, positive functions [14].

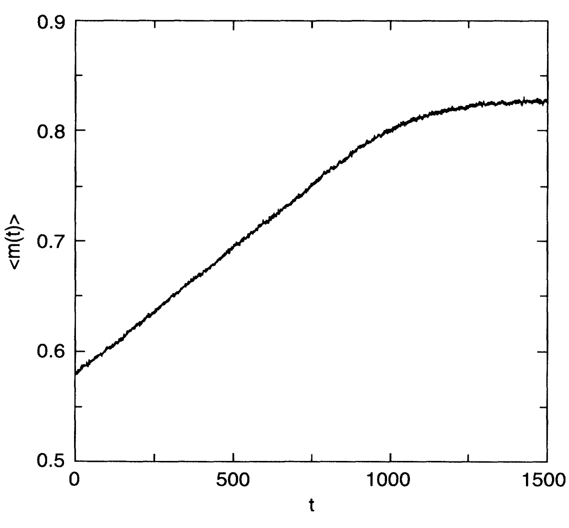

FIG. 5. Decay of droplet area as a function of time. A 90×90 lattice is first equilibrated in the positively magnetized ordered phase with periodic boundary conditions, and then a circular domain of spins of a given total area is inverted at time $t=0$. The return of average total *signed* magnetization $\langle m(t)\rangle\equiv(1/N)\sum_i\langle \mathrm{sgn}(y_i^t)\rangle$ to its equilibrium value, in this case $\langle m(\infty)\rangle\simeq0.831$, is shown as a function of time. The plot is an average of 200 such droplet events with $g=0.223$.

The factorization carried out in (20) is not possible for an arbitrary Langevin equation. The most general formulation of a relaxational model may be described by (1), as we indicated above. In our case, the i,j in (1) index the modes θ. The only contribution to M_{ij} at long wavelengths is diagonal in i,j, because the coupling among distinct angular modes is irrelevant at long wavelengths, as is apparent from (15). The diagonal character of M_{ij} in this case may be related to the geometrical nature of the dynamics; we have no general understanding of when it is to be anticipated.

C. Droplet nucleation

We have argued in the previous subsection that the (bulk) behavior of the domain walls is determined by an Ising free energy. Once *sufficiently large* droplets are created, we expect their evolution to be determined by the Langevin dynamics of the domain walls which bound a droplet. For a *Hamiltonian* Ising model at nonzero temperature, ergodicity *guarantees* the creation of arbitrarily large droplets; however, we cannot in general expect dynamical models of the form we are studying to be ergodic.

So far, we can only address the emergence of arbitrarily large droplets empirically; we have no analytic argument for the occurrence of droplets sufficiently large that the Langevin dynamics will govern the evolution of their domain walls. The best we can do is to show numerically that there is no cutoff in droplet size up to the largest spontaneously generated droplets we can obtain within reasonable simulation times. Unfortunately, the probability of a droplet of area D is expected to be exponentially rare (more precisely, scaling as $\exp\{-\tau_{\mathrm{av}}D^{1/2}\}$). Both for our model and for *equilibrium* Ising models where droplet properties have been established by rigorous argument [15], finite-size effects are so severe that we are unable to obtain spontaneously generated droplets sufficiently large that they fall into a regime where the Langevin argument clearly applies. In particular, we are unable to obtain spontaneously generated droplets of the size we earlier artificially introduced in order to examine the dynamics of droplet decay. A direct comparison of our model and, say, the usual nearest-neighbor ferromagnetically coupled Ising model is further hampered by the fact that there is no particular reason why the effective local couplings for our model ought to be either nearest neighbor or exclusively ferromagnetic, and no expectation that any finite-size corrections should be the same for both models. In fact, our model does exhibit a significant amount of *short-range* antiferromagnetic order, in addition to its long-range ferromagnetic order.

Consequently, we rely on the *smoothness* of the decay in frequency of droplets as a function of their size. We measured the relative frequency of occurrence of droplets, which we defined for numerical purposes as connected clusters of antiphase spins [16]. If there existed a cutoff in droplet size, we would expect our curve to be

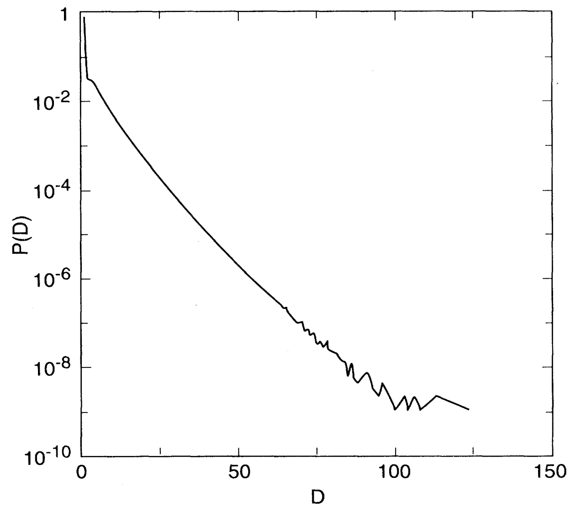

FIG. 6. Histogram of probability of *connected* droplets of area D. At each iteration of a 30×30 lattice, all connected droplets are found and counted. The fraction of the connected droplets that have area D is plotted as a function of D on this semilogarithmic plot. [A "connected droplet" is defined here as a set of D lattice sites each with sgn(y) opposite to that of the majority phase, which are connected by nearest-neighbor bonds and the droplet is surrounded by sites with sgn(y) the same as the majority phase.] These data are for $g=0.223$.

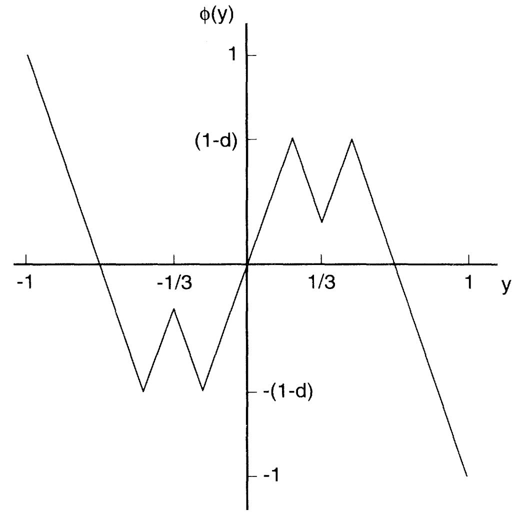

FIG. 7. The "folded" map function.

sharply truncated. Instead, *the observed droplet frequency decays smoothly into the noise* (Fig. 6). We have also examined equilibrium nearest-neighbor Ising models, and find similar behavior in droplet frequency; in neither case have we been able to obtain droplets of sufficient size to observe $\exp\{-\tau_{\rm av}D^{\mu}\}$ decay with $\mu=\frac{1}{2}$. Rather μ appears to vary continuously from near 0 to near unity as g is increased from g_c in our coupled-map array, or T is decreased from T_c in the equilibrium Ising model.

OTHER PHASES

In the preceding sections, we have presented evidence that our deterministic array has an ordered phase indistinguishable, at long wavelengths, from that of a 2D Ising model. For the map we have studied above, we believe we understand the entire phase diagram as a function of coupling, g; however, if we alter this map, other kinds of phases are possible. One possible modification is to change the map function $\phi(y)$ which enters in Eq. (9) as indicated in Fig. 7, where we have "folded" the peaks of the simple tent map function of Fig. 1 by some distance d, keeping the slope $|d\phi/dy|=3$. Provided d is set to some value d_0 within the interval $\frac{1}{3}<d_0<\frac{1}{2}$, the uncoupled map for a single site, which was ergodic for $d=0$, has two disjoint *chaotic* attractors, related by sign inversion. At small coupling our array with $d=0$ has a single paramagnetic attractor; for $d=d_0$, the array has 2^N distinct *chaotic* attractors, corresponding to the two possible symmetry-related states for each N independent sites. Thus one has attractors with complete ferromagnetic or antiferromagnetic order, as well as attractors with static domain walls in any possible pattern. In this regime the domain walls are perfectly static and droplets are never nucleated within any of the domains.

Based upon our identification of defect dynamics as a basic characterization of the phases of the kind of model we are studying, we may speculate on what other phases might occur. First, there could be a phase where the large-scale domain-wall dynamics remain the same as discussed above, so all droplets decay away, but the nucleation of large droplets does not occur. In this phase there would still be only two attractors for the dynamics, but the fluctuations within the attractors would be limited to only small droplets or just fluctuations in the magnitudes of the y_i. Second, the domain walls could undergo a sort of roughening transition to a regime where a long straight domain wall is stationary. Then the system would have more attractors with such walls present. Once a straight domain wall is stationary, one can investigate the dynamics of other structures, such as the corner of a large domain, a step in an otherwise straight domain wall, or the intersection of two domain walls. Phases, which could be distinguished by the mobility or stationarity of these defects, might exist. Once some sort of domain-wall corner or intersection is stationary, the number of distinct attractors will presumably grow as $\exp(sN)$, where N is the number of sites and s is a sort of entropy density. Thus we could characterize a phase with s, which will take on the value $s=\ln 2$ in the phase with 2^N attractors discussed above. An investigation of these possible phases and the phase transitions between them, for example as one varies d and g, would be of much interest.

EXTERNAL NOISE

A well-known, but so far unproven, hypothesis maintains that stochastic cellular automata in *one* dimension are, in their long-wavelength properties, Ising models, provided they are ergodic, and homogeneous in space and time [17]. Our own conjecture represents more than a trivial extension of this hypothesis, since the 2D Ising model displays a number of important features not present in one dimension. In particular, the 1D model has only a zero-temperature phase transition; it is paramagnetic, except at zero temperature where it has long-range order. In contrast, the 2D Ising model has a range of temperatures, below T_c, for which it has long-range order, and for which the defects play the dominant role in determining the correlations. It is on this more subtle regime, absent in one-dimension, that we have concentrated here.

Nevertheless, our arguments suggest that an analogous hypothesis is probably valid in two dimensions. A stochastic, local noise that reversed the signs of the spins would create droplets large enough that the Langevin dynamics (15) ought to apply. These droplets would be sufficient to generate 2D Ising correlations in the ordered phase, even when the deterministic (noise-free) system does not spontaneously create large droplets [18].

CONCLUSIONS

We have argued that symmetry considerations enable us to understand the long-wavelength properties of a class of locally coupled, deterministic, nonequilibrium system. We have suggested that a rich phase structure may emerge for the deterministic dynamics of coupled maps with Ising symmetry. While microscopically irreversible, the dynamics of Eqs. (8) and (9) recovers a sufficient degree of reversibility at large length scales that it may be viewed, in a coarse-grained sense, as satisfying detailed balance. We have proposed that this large-scale reversibility may be understood in terms of the relevance or irrelevance of corrections to a Hamiltonian form for the dynamics. Whether and when such a situation occurs in more complex situations is a subject for future study.

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[13] For example, algorithms for finding the interface yield identical scaling behavior regardless of their treatment of overhangs.
[14] We may increase the anisotropy in the stiffness by augmenting the "fold," d, described later; near a critical value the ratio of $\sigma(\pi/4)$ to $\sigma(0)$ apparently diverges. Such a divergence is also seen in equilibrium Ising models at $T=0$.
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[17] See, e.g., A. L. Toom, in *Multicomponent Random Systems,* edited by R. L. Dubroshin, Advances in Probability Vol. 6 (Dekker, New York, 1980), and references contained therein.
[18] To generate a droplet dynamics which deviates from that of the equilibrium Ising model, Bennett and Grinstein in Ref. [10] have introduced in their stochastic model a spatial anisotropy such that domain walls move in a preferred direction. The effect of the domain-wall motion is to increase the rate at which a droplet is annihilated, so that a droplet disappears in a time proportional to its radius, rather than its area as we find here.